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TRABALHO DE CONCLUSÃO DE CURSO

Código de acesso aleatório quântico em canais ruidosos

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Resumo

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Código de acesso aleatório quântico em canais ruidosos

por Rafael Alves da Silva

O código de acesso aleatório (CAA) é um protocolo de comunicação particularmente útil quando a comunicação entre as partes é restrita. Neste trabalho, nos baseamos em trabalhos anteriores que demonstraram que, na ausência de ruídos, o código de acesso aleatório quântico (CAAQ) supera o código de acesso aleatório clássico (CAAC). Aqui, nós investigamos os efeitos de canais ruidosos sobre a performance do CAAQ quando comparado a sua versão clássica correspondente.

Palavras-chave: comunicação quântica, canais quânticos ruidosos, código de acesso aleatório.

Quantum Random Access Code in Noisy Channels

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The random access code (RAC) is a communication protocol particularly useful when the communication between parties is restricted. In this work we built upon works that have previously proven quantum random access code (QRAC), in the absence of noise, to be more advantageous than classical random access code (CRAC). Here we investigate the effects of noisy channel on the QRAC performance when compared to its classical counterpart.

Keywords: Quantum communication, noisy quantum channels, random access code.

I. INTRODUCTION

Many aspects that distinguish a quantum system from a classical one can be used as valuable resources in information theory. Quantum information and quantum communication, for instance, study how to use quantum systems to implement information processing, *e.g.*, quantum algorithms (Deutsch [1], Shor [2], Grover [3], etc.); Communication protocols (super dense code [4], teleportation [5], quantum key distribution [7], etc.), and quantum simulation . In quantum communication, particularly, one uses quantum resources such as entanglement and superposition to enhance information transmission beyond classical limitations [16, 19].

An important example of communication protocol is the random access code (RAC) [8]. This protocol is especially suited when there is restriction on the size of the message. In a RAC protocol, one party, Bob, aims to access an arbitrary subset from a data set held by a second party, Alice, using a channel with restricted information capacity. In this scenario the information cannot be sent deterministically because the amount of information is larger than the channel capacity. However it is possible to use strategies of message encoding and decoding that improve Bob's probability of accessing correctly the information. Moreover, the RAC protocols can be used for network coding [10], semi-device independent key distribution [11], reduction of communication complexity [8], etc.

It was shown that using quantum RAC the probability of success can be improved for different set of information held by Alice and the channel capacity [12]. However, to the best of our knowledge, many of works that have dealt with RAC have not considered noise channels. Even with our best effort, we cannot isolate a system from its surroundings completely [13]. Based on that, it becomes evident in order to construct a realistic RAC protocol one has to account for the noise introduced in the system due to its environment.

In this work, we investigate how system-environment interactions affect Bob's probability of accessing a given subset of the information held by Alice.

In section (II) we formally introduced the random access code protocol, its main features, and showed the main differences between quantum RAC and classical RAC. In section (III) we presented a basic overview of theory of open quantum system and how it relates to concept of noisy quantum channels, and we also introduce the three noisy quantum channels studied in this work: depolarizing, amplitude damping, and dephasing channels. In section (IV), we introduce, in passing, the concepts of Von Neumann entropy and fidelity, and how they relate to the present work. In section (V) we applied the concepts laid out in the previous section to study the behaviour and performance of quantum RAC in those noisy channels. In section (VI) we summarize the conclusions and discuss new perspectives for futher studies.

II. RANDOM ACCESS CODE

In a RAC, Bob aims to access an arbitrary subset of a set of information held by Alice, with average probability of success P. Even though the communication between parties is restricted, they can improve the probability of success by improving the communication strategy [12].

Before proceeding, we shall introduce a very useful shorthand notation for both classical and quantum RAC. Suppose Alice has a n-letter string (word), with each letter being encoded in d-level systems, *i.e.*, $X = x_0 \ldots x_{n-1}$, where $x_i \in \{0, \ldots, d-1\}$. Bob, upon receiving Alice's string, wishes to access an arbitrary letter x_j with average success probability $P_{n,d}$. This

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can be summarized as,

$$n^{(d)} \xrightarrow{P_{n,d}} 1. \tag{1}$$

A. Classical Random Access Code

A RAC is classified as *classical* when the information is encoded in classical d-level systems (see figure 1).

The simplest example of CRAC is in the form of $2^2 \to 1$. The best strategy, in this case, is the one Alice always sends to Bob the value of her first letter (x_0). Whenever Bob is interested in x_0 the probability is $P_0 = 1$, but if he is interested in x_1 he is forced to guess. Since x_1 can take up two values, 0 or 1, Bob has a probability $P_1 = 1/2$ of success. Thus, Bob's average probability of success in this scenario is $P^C = (P_0 + P_1)/2 = 3/4$.

A case more general than the one mentioned above is the scenario in which x_0 and x_1 are encoded in d-level system, i.e $2^d \to 1$. In analogy to the previous case, Bob's average probability of success is given by [12],

$$P^C = \frac{1}{2}\left(1 + \frac{1}{d}\right). \tag{2}$$

It should however be emphasized that although there are other families of RACs for which $n > 2$, that is, the cases in which Alice has words with more than two letters ($X = x_0 x_1 ... x_{n-1}$), we are only considering in this work the particular case of two-lettered words ($n = 2$). From now on, when we reference the word **RAC** (or its derivatives) we are actually referencing the $2^d \to 1$ RAC.

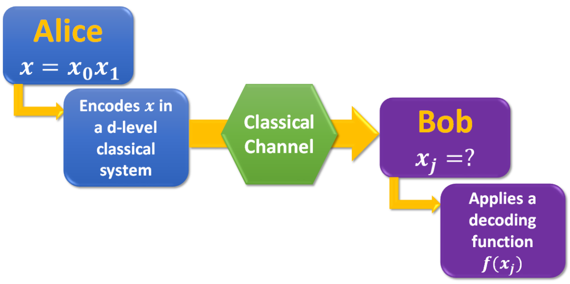

Figure 1. Schematics of a classical RAC protocol.

B. Quantum Random Access Code

In the quantum RAC (QRAC), whenever Alice has a two-letter word $X = x_0 x_1$, she encodes it in a quantum d-level system [12] (see figure 2). A state for such system can be constructed using two mutually unbiased bases (MUBs), namely, the computational basis $\{|l\rangle\}$ and the Fourier basis $|e_l\rangle = (1/\sqrt{d}) \sum_{k=0}^{d-1} \omega^{kl} |k\rangle$, with $\omega = \exp\{2\pi i/d\}$. These bases together allow us to encode the state as:

$$|\psi_{x_0 x_1}\rangle = N(|x_0\rangle + |e_{x_1}\rangle), \tag{3}$$

where $N = 1/\sqrt{2 + (2/\sqrt{d})}$ is the normalization constant.

Whenever Bob is interested in x_0, he performs a measurement in the computational basis $|l\rangle$,

$$P_0(l) = |\langle l|\psi_{x_0 x_1}\rangle|^2 = N^2 \left| \delta_{l,x_0} + \frac{\omega^{x_1(l-x_0)}}{\sqrt{d}} \right|^2, \tag{4}$$

and when he is interested in x_1, he measures in the Fourier basis $|e_l\rangle$,

$$P_1(l) = |\langle e_l|\psi_{x_0 x_1}\rangle|^2 = N^2 \left| \omega^{-x_0 x_1} \delta_{l,x_1} + \frac{\omega^{-l x_0}}{\sqrt{d}} \right|^2, \tag{5}$$

which gives an overall average probability of success of

$$P^Q = \frac{1}{2}\left(1 + \frac{1}{\sqrt{d}}\right). \tag{6}$$

It is worth mentioning that, because the states $|\psi_{x_0 x_1}\rangle$ are symmetrically distributed in the Bloch (hyper)sphere (see figure (3), where this is illustrated for $d = 2$), the probabilities of all outcomes are the same (equal to P^Q) regardless of which string $x = x_0 x_1$ Alice has or which letter Bob is interested in [12]. As we will see in sections III and V, this is not always the case for noisy channels. For instance, the dephasing and amplitude damping destroy this "symmetry", thus making some outcomes more likely than others.

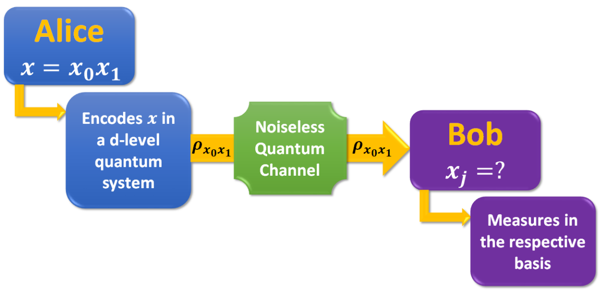

Figure 2. Schematics of a quantum RAC protocol through a noiseless channel.

We conclude by comparing (2) and (6) that, in the absence of noise, Bob's probability of success in the quantum case is always greater than the classical one ($P^Q/P^C > 1\ \forall d$).

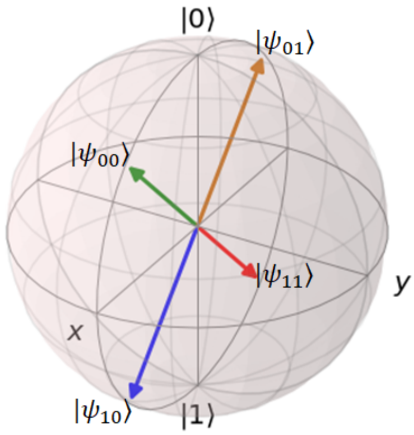

Figure 3. Illustration of the distribution of the states $|\psi_{x_0x_1}\rangle$ on the Bloch sphere when $d=2$. The states are symmetrically distributed and form a square of whose each of them is a vertex.

III. OPEN QUANTUM SYSTEMS AND DYNAMICAL MAPS

Even though, the time evolution of a closed system can be described in terms of unitary evolution, the dynamics of an open quantum system generally cannot [13]. In the simplest scenario, one has an open quantum system, S, which interacts with another quantum system, the environment E [14] (see figure 4).

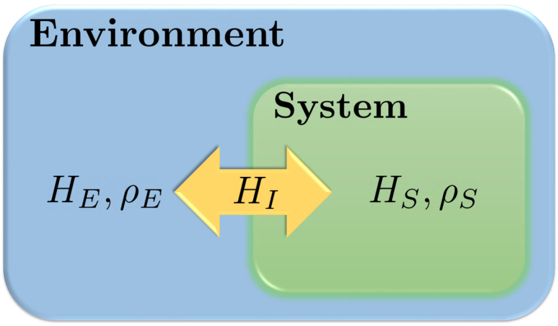

Figure 4. Representation of an open quantum system.

Therefore, the open system S is a subsystem of a larger system $S+E$. Because $S+E$ is a closed system, its dynamics can be in general described in terms of unitary operators. On the other hand, the subsystem S cannot [13]. The Hamiltonian for the total system $S+E$ is,

$$H = H_S \otimes I_E + I_S \otimes H_E + H_I, \tag{7}$$

where H_S is the self-Hamiltonian of S, H_E is the free-Hamiltonian of E, and H_I is the interaction Hamiltonian, and I_S and I_E are the identity operators for S and E, respectively. The time evolution operator of $S+E$ is given by $U(t,t_0=0)=\exp\{-iHt\}$ in natural units. Let $\rho(0)$ be the density operator describing $S+E$ at an initial time $t_0=0$. At time $t>t_0$ the state of the whole system will be,

$$\rho(t) = U\rho(0)U^\dagger. \tag{8}$$

Now assuming that the state $\rho(0)$ is initially uncorrelated, that is,

$$\rho(0) = \rho_S(0) \otimes \rho_E(0), \tag{9}$$

the evolution of the system $S+E$ from $t_0=0$ to $t>0$ may be written as

$$\rho(t) = U\rho_S(0) \otimes \rho_E(0)U^\dagger. \tag{10}$$

Let $\{|\nu\rangle\}$ be a basis for E. We assume, without loss of generality, that the initial state of E is $|0\rangle$, hence $\rho_E(0)=|0\rangle\langle 0|$. Therefore, equation (10) becomes

$$\rho(t) = U\left(\rho_S(0) \otimes |0\rangle\langle 0|\right)U^\dagger. \tag{11}$$

The state of the system S is found by performing a partial trace over the degrees of freedom of the environment $\rho_S(t)=Tr_E[\rho(t)]$, which can be written explicitly as,

$$\begin{aligned} \rho_S(t) &= Tr_E[U\left(\rho_S(0)\otimes|0\rangle\langle 0|\right)U^\dagger] \\ &= \sum_\nu \langle\nu|U|0\rangle\rho_S(0)\langle 0|U^\dagger|\nu\rangle \\ &= \sum_\nu K_\nu(t)\rho_S(0)K_\nu^\dagger(t), \end{aligned} \tag{12}$$

where $K_\nu=\langle\nu|U|0\rangle$ are the so-called Kraus operators, which satisfy $\sum_\nu K_\nu K_\nu^\dagger = I$, a property that guarantees that $Tr[\rho_S(t)]=1\ \forall t$. Besides, the time evolution as presented in equation (12) keeps the positive semidefinite character of $\rho_S(0)$, and, together with the preservation of the trace, ensures that $\rho_S(t)$ is still a (physical) density matrix [17].

The time evolution of $\rho_S(0)$ from $t=0$ to $t>0$ can also be performed by employing a super-operator $V(t)$

$$\rho_S(0) \to \rho_S(t) = V(t)\rho_S(0). \tag{13}$$

If we consider both ρ_E and t as being fixed, equation (13) represents the following map,

$$V(t) : \mathcal{S}(\mathcal{H}_S) \to \mathcal{S}(\mathcal{H}_S), \tag{14}$$

which is called a dynamical map (see figure 5), where $\mathcal{S}(\mathcal{H}_S)$ is the space of density matrices of the subsystem S [13]. For every $t\geq 0$ there corresponds a map dynamical $V(t)$, which together form a family of dynamical maps whose only parameter is t, and which satisfies $V(0)=I$, where I is the identity operator.

$$\begin{tikzcd}
\rho(0) = \rho_S(0) \otimes \rho_E(0) \arrow[r, "\text{Unitary evolution}"] \arrow[d, "Tr_E"'] & \rho(t) = U[\rho_S(0) \otimes \rho_E(0)]U^\dagger \arrow[d, "Tr_E"'] \\
\rho_S(0) \arrow[r, "\text{Dynamical map} \quad V(t)"] & \rho_S(t)
\end{tikzcd}$$

Figure 5. The conceptual idea behind a dynamical map is illustrated in the form of a commutative diagram.

A complete positive and trace-preserving (CPTP) linear map in the form of equation (12) is called a quantum channel. The word "channel" comes from communication theory. The analogy is based on the fact that one party (Alice) transmits the state ρ through a communication path to another party (Bob), who receives the (generally) modified state ρ' [18]. Noise is introduced into the system S as a result of its interaction with the environment E. In other words, part of the information contained in the state ρ is lost due to this interaction [19].

When we are dealing with Markovian channels, for which memory effects may be neglected, there exist certain super-operators $\mathcal{L}$, which allow us to write $V(t)$ in an exponential form [13],

$$V(t) = \exp\{\mathcal{L}t\}. \tag{15}$$

Notice that by taking the derivative of $\rho_S(t)$, and taking equation (15) into account it yields,

$$\begin{aligned}
\frac{d\rho_s(t)}{dt} &= \frac{d[V(t)\rho_s(0)]}{dt} \\
&= \frac{d[\exp\{\mathcal{L}t\}]}{dt}\rho_S(0) \\
&= \mathcal{L}V(t)\rho_S(0) = \mathcal{L}\rho_S(t).
\end{aligned} \tag{16}$$

Equation (16) is called Markov quantum master equation. The super-operator $\mathcal{L}$ is called the Liouvillian in analogy to Liouville equation from classical mechanics. Although equation (16) informs us that the markovian evolution a density matrix is governed by a first order differential equation, it does not guarantee that all possible solutions will be physical, *i.e.*, positive semidefinite operators [9]. In the case of markovian processes, Lindblad [21] showed that in order for the equation (16) to give physical solutions, it must have the following form,

$$\dot{\rho}_S = -i[H_S, \rho_S] + \sum_j \left[2A_j\rho_S A_j^\dagger - \left\{A_j^\dagger A_j, \rho_S\right\}\right], \tag{17}$$

where A_j are the so-called Lindblad operators. In the next subsections we will introduce the noisy channels that we have studied.

A. Depolarizing Channel

The depolarizing channel is used to analyse experimental setups where quantum systems may be lost or when one has non-ideal detectors. For qudits (d-level quantum systems), this channel can be described as follows: ρ has a probability p of being replaced with a completely mixed state, I/d, otherwise it remains unchanged [19]. The corresponding map is,

$$\rho_S(t) = \frac{pI}{d} + (1-p)\rho_S(0), \tag{18}$$

where the probabilities are given, as a function time, by $p = 1 - e^{-\Gamma t}$, where Γ is the depolarizing decay parameter that depends on the system-environment coupling strength.

One important aspect of the depolarizing channel is its symmetry (see figure (6)). This property guarantees, for example, that Bob has the same probability of success either if he measures in the computational basis or in the Fourier basis.

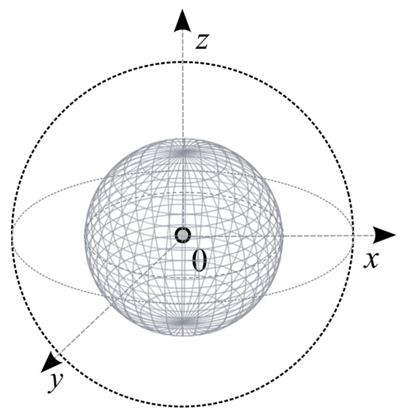

Figure 6. Representation on the Bloch Sphere of a depolarizing channel acting on a qubit. At $t = t_0$ the states- which are pure - lie on the surface of the sphere (represented by the dotted lines). As time passes and the states become mixed under the influence of noise, they now lie inside the old Bloch sphere as if it had shrunk. At a long enough time the state under the influence of this type of decoherence will evolve to a completely mixed state represented by a single point at the center of the old Bloch sphere [16].

B. Dephasing Channel

The dephasing channel is a kind of quantum channel in which the quantum information of the system is lost but its energy eigenstates remain constant in time (there is no thermalization) [18]. Dephasing is characterized mathematically by the exponential decay of the off-diagonal elements of the density matrix and the preservation of its diagonal elements.

The master equation for a system composed of a single qudit with degenerate energy transition, *e.g.*, truncaded harmonic oscillator, that undergoes dephasing due to its interaction with a zero-temperature environment is given by [6, 9],

$$\dot{\rho}_S = \Gamma \left[2a^\dagger a \rho_S a^\dagger a - \left\{ (a^\dagger a)^2, \rho_S \right\} \right], \tag{19}$$

where Γ is the dephasing system-environment coupling constant, and a and $a^\dagger$ are the annihilation and creation operators, respectively. By solving (19) one finds that the elements of the initial density matrix $\rho_S(0)$ will evolve as,

$$\langle n|\rho_S(t)|m\rangle = (1-p)^{(n-m)^2} \langle n|\rho_S(0)|m\rangle , \tag{20}$$

where $(1-p) = e^{-\Gamma t}$.

An interesting feature of this channel is the fact that it deforms the Bloch sphere unevenly (see figure (7)), but in a manner that its poles remain fixed, which means that measurements in the computational basis yield always the same probability while in the Fourier basis the probability changes with time.

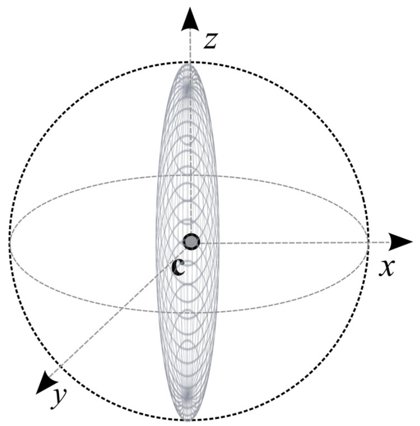

Figure 7. Representation on the Bloch sphere of a dephasing channel acting on a qubit. The situation here is analogous to the depolarizing map with the exception that dephasing map "shrinks" the Bloch sphere unevenly. [16]

C. Amplitude Damping

Amplitude damping is the process by which a system (atom, spin, harmonic oscillator, etc.) undergoes a transition from a higher state of energy to a lower state of energy by losing energy to the environment [20]. It captures the dissipating nature of the system-environment interaction, as it not only destroys coherence but also causes the system to lose energy by driving it towards the ground state [17, 18, 20]. Geometrically, this map not only shrinks the Bloch sphere into an ellipsoid but also moves its center along the z axis (see figure (8)) and is, because of this, a non-unital channel, *i.e.*, a channel that does not map the identity operator to itself [16].

The scenario being considered for this work is the particular case of a system composed of a single qudit interacting solely with a zero-temperature environment of electromagnetic-field modes [20]. The evolution of such a system is governed by the following master equation,

$$\dot{\rho}_S = \Gamma \left[2a\rho_S a^\dagger - \left\{ a^\dagger a, \rho_S \right\} \right], \tag{21}$$

where Γ is the system-environment coupling constant, and a and $a^\dagger$ are the annihilation and creation operators, respectively.

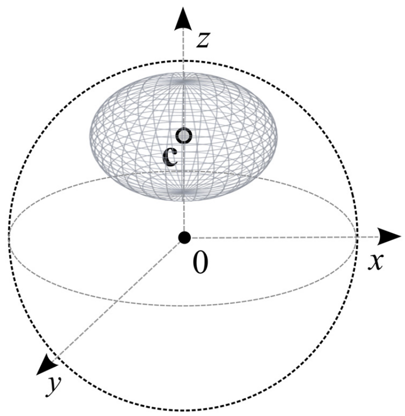

Figure 8. Representation, on the Bloch sphere, of the amplitude-damping channel acting on a qubit. This map shrinks the Bloch sphere in the two directions of the equatorial plane, and also moves the center of resultant ellipsoid towards the north pole, which represents the fundamental state. [16].

IV. INFORMATION LOSS: A QUALITATIVE APPROACH

In this section, we briefly introduce two important concepts that will help us understand better why the QRAC protocols behave the way they do in noisy channels: the von Neumann entropy and the input-output fidelity.

A. Von Neumann Entropy

The Von Neumann entropy (S_{VN}) plays an essential role in quantum information theory as Shannon entropy does in classical information theory [16]. As an extension of Shannon entropy to the quantum theory, S_{VN} is a quantifier of uncertainty related to a quantum state. It is defined mathematically as:

$$S_{VN} = -Tr[\rho \log_2 \rho] = -\sum_i \lambda_i \log_2 \lambda_i, \qquad (22)$$

where λ_i are the eigenvalues of ρ. For pure states, $S_{VN} = 0$, and for a maximally mixed state, $S_{VN} = \log_2 d$, where d is the dimension of the qudit.

As qudits undergo decoherence through a noisy quantum channel, the von Neumann entropy related to its state changes, usually increases. This means that less information can be obtained from a measurement, which, in the context of this work, explains why QRAC performance decreases when we are dealing with noisy channels.

B. Fidelity

A sometimes very useful measure of distance between quantum states is the fidelity. It can be thought as an overlap or even an inner product of two states [19, 20]. Given two states ρ and σ, the fidelity is defined as

$$F = tr\sqrt{\rho^{1/2}\sigma\rho^{1/2}}. \qquad (23)$$

A particular type of fidelity is the input-output fidelity, which measures the dissimilarity between the input and output states of a quantum map [23]. Consider an initially pure state $\rho(0) = |\psi\rangle\langle\psi|$ as the input of a certain channel, and the state $\rho(t)$ as the output, where $t > 0$. The input-output fidelity is given by,

$$F_{IO} = \sqrt{\langle\psi|\rho(t)|\psi\rangle}. \qquad (24)$$

Thus, in the context of this work, F_{IO} describes the relation (and contrast) between Alice's pure input state $\rho(0)$ and the (generally) mixed output state $\rho(t)$ received by Bob.

V. QRAC WITH NOISE

In absence of noise, one can infer, based on equations (2) and (6), that Bob's probability of success is always higher with QRAC than with CRAC, *i.e.*, $P^Q > P^C$ for any dimension d.

However, when dealing with noisy channels, there is not any guarantee that P^Q will indefinitely remain greater than P^C as time passes. As a matter of fact, the type of dynamics plays an important role. The combination of factors such as type of noise, coupling constant and dimension of the system will determine whether or not the ratio P^Q/P^C will remain greater than one and for how long. Figure 9 shows a scheme illustrating how we will consider the noise in the QRAC $2^d \to 1$.

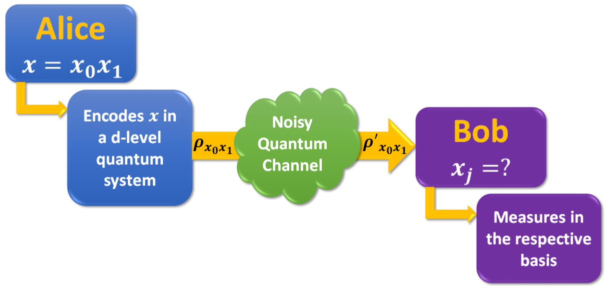

Figure 9. Schematics of a quantum RAC protocol through a noisy channel.

Let us now consider the d^2 possible states $\rho_{x_0x_1} = |\psi_{x_0x_1}\rangle\langle\psi_{x_0x_1}|$ Alice may have at a time $t = 0$. At a time $t > 0$, after going through a noisy channel, the states will be mapped to,

$$\rho'_{x_0x_1} = \rho_{x_0x_1}(t) = \sum_\nu K_\nu(t)\rho_{x_0x_1}K^\dagger_\nu(t), \qquad (25)$$

where the Kraus operators K_ν depend on the type of channel. We can use the result of (25) to compute Bob's time-dependent probability of success

$$P^Q(t) = \frac{1}{2d^2}\sum_{x_0,x_1=0}^{d-1} \mathrm{Tr}\{\rho_{x_0x_1}(t)[|x_0\rangle\langle x_0| + |e_{x_1}\rangle\langle e_{x_1}|]\}. \qquad (26)$$

We simulated the impact of depolarizing, amplitude damping, and dephasing channels on Alice's input states and how these channels affected Bob's probability of success. First, we looked at how the exposure to noise sources affects the ratio P^Q/P^C as a function of time. The overall behaviour is illustrated on figure (10). Contrary to the noiseless case, where QRAC is always superior to CRAC at any instant of time and for any dimension, when one considers noise the scenario changes rather drastically. Although the curves on the graphs of figure (10) have different rates of decay, depending on the kind of dynamic and the dimension, in all cases the P^Q/P^C ratio will eventually drop below one, from which point CRAC becomes superior to QRAC. Another interesting fact is that, for any of the three dynamics, as the dimension d increases, it takes increasingly shorter amounts of time for the QRAC to become less advantageous than CRAC. This can be seen, for example, with the blue curves ($d = 2$), which start at $t = 0$ at the smallest P^Q/P^C ratio when compared to the other curves, but are able to sustain $P^Q/P^C > 1$ for the longest time.

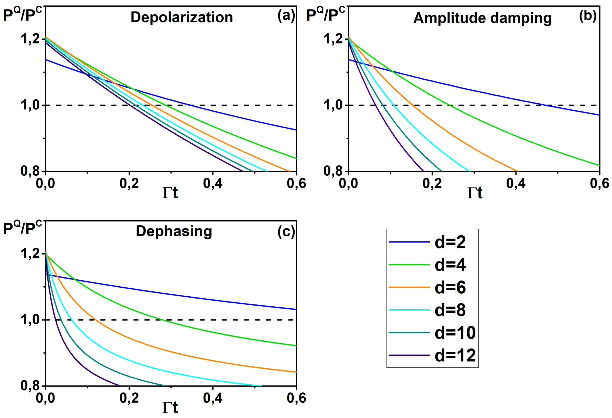

Figure 10. Each graph shows how the ratio between quantum and classical average probabilities of success as function of time for each of the three quantum channels.

The graphs on figure (11) show the behaviour of the input-output fidelity (F_{IO}) as a function of time. A striking feature of the fidelity's curves is how much they resemble the P^Q/P^C's ones. This resemblance indicates that the decay of F_{IO} is related to the decay in P^Q/P^C. One interpretation to this is the fact that as the state $\rho(t)$ received by Bob becomes more dissimilar to the state $\rho(0)$ prepared by Alice, becoming harder for Bob to succeed, therefore P^Q/P^C also decreases.

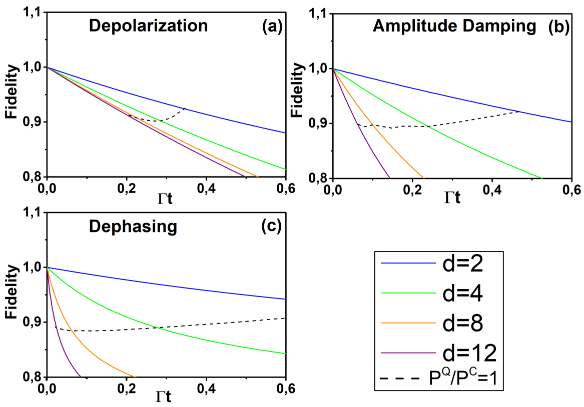

Figure 11. Each graph shows how the input-output fidelity chances as function of time for each of the three quantum channels.

The dimension d also plays a significant role on Bob's chances of success. As a matter of fact, the higher d gets the poorer Bob's performance is. This effect can be seen in figures (12), where we explored the influence of increasing the dimension while keeping time fixed (*i.e.*, considering the same elapsed time). One striking feature of noisy channel is the fact that, even for dimensions as low as $d = 2$, it is only a matter of time for QRAC advantage over CRAC to completely vanish. On the flip side, for any value of $\Gamma t > 0$, no matter how low, if you increase the dimension enough, it will eventually lead to QRAC performing poorly. Further analysis of these graphs indicates that high-dimensional versions of QRAC are sensible to the dephasing and amplitude damping channels the most, although, for the dephasing channel, they are still able to keep higher values of P^Q/P^C than for other dynamics. For the depolarizing channel, on the other hand, QRAC performance seems to be more tolerant to high dimensions, at least for values of $\Gamma t < 0.69$.

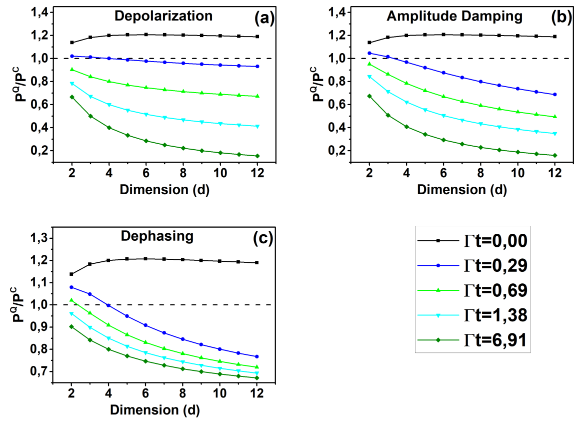

Figure 12. This plot shows the ratio between quantum and classical average probabilities of success as function of dimension d for all three channels in different time spans.

In the graphs in figure (13) we focused on the time (life span), fidelity and entropy specifically when QRAC becomes less advantageous than CRAC ($P^Q/P^C = 1$) as functions of the dimension. In graph (a) we see that the dimension has major impact on how fast P^Q becomes less than P^C. In this regard, QRAC going through dephasing is affected the most by an increase in the dimension, which is reasonable considering that for this channel the coherence decay is proportional to $e^{(n-m)^2\Gamma t}$, according to equation (20). Nevertheless, for dimension $d = 2$ and $d = 3$, this channel is the best among the three in terms of life span, and it only becomes the worst after $d = 6$. In the case of the amplitude damping channel, the effects of high dimensions is considerable, although less pronounced than for dephasing. QRAC in the depolarizing channel presents, at dimension $d = 2$, the shortest life span of the three, but for $d > 4$ is able to perform better than in the other channels. In the graph (c) we show the behaviour of the entropy for when $P^Q/P^C = 1$. The entropy grows more with the dimension for the depolarizing channel than for the other dynamics and do not ceases to grow (at least not for $d \leq 12$). If we look back at the depolarizing curve in graph (a) of same figure, we see a steady (almost linear) decrease in the time for which $P^Q/P^C = 1$. For dephasing and amplitude damping dynamics, we see a relatively high increase of entropy with the dimension from $d = 2$ up to $d = 6$, when, finally, its growth starts to slow down. Again analysing graph (a), we notice that the time for which $P^Q/P^C = 1$ decreases relatively fast from $d = 2$ to $d = 6$, and then its decrease rate starts to diminish. In our interpretation, this indicates that the increase in entropy relates (although not directly proportionally) to the decrease of QRAC performance, resulting lower life span. The reason for this, we argue, is the fact that the faster the entropy (disorder) grows, the faster the amount of information that can obtained from state decline, which in turn lowers Bob's probability of success because part of the information Alice encoded in the state is lost.

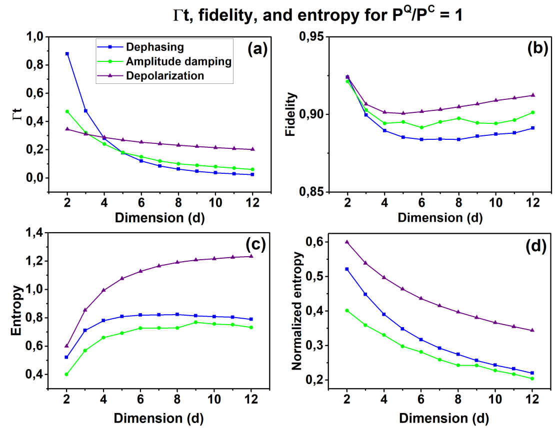

Figure 13. The plot shows: (a) time when P^Q becomes less than P^C, (b) Fidelity for $P^Q/P^C = 1$, (c) entropy, and (d) the normalized entropy for $P^Q/P^C = 1$ as functions of the dimension d.

VI. CONCLUSION

In this work we reviewed the concept of random access code in both its classical and quantum versions. Based on previous works, we presented a generalization of QRAC and CRAC for d dimensions, which led us to the conclusion that, in the absence of noise, QRAC always outperforms CRAC regardless of the dimension. We built upon this work by incorporating the theory of open quantum systems to understand how a noisy channel affects the performance of QRAC. We showed, for all the three noisy channels, the decrease in QRAC advantage over CRAC using fidelity and entropy. We found that, for a given channel, the magnitude of the decrease in performance depends on both dimension and elapsed time.

In future works, we intent to continue our investigation focusing mainly on the possibility of optimizing the quantum RAC for each of the noisy channels studied in the present work, determining whether or not the effects of quantum decoherence can be at least partially mitigated. In order to perform this optimization, we will apply semidefinite programming (SDP) to look for better encoding and decoding strategies.

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